

Asymptotic aspects of Cayley graphs

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Abstract

Arising from complete Cayley graphs Γ_n of odd cyclic groups \mathbf{Z}_n , an asymptotic approach is presented on connected labeled graphs whose vertices are labeled via equally-multicolored copies of K_4 in Γ_n with adjacency of any two such vertices whenever they are represented by copies of K_4 in Γ_n sharing two equally-multicolored triangles. In fact, these connected labeled graphs are shown to form a family of graphs of largest degree 6 and diameter asymptotically of order $|V|^{1/3}$, properties shared by the initial member of a collection of families of Cayley graphs of degree $2m \geq 6$ with diameter asymptotically of order $|V|^{1/m}$, where $3 \leq m \in \mathbf{Z}$.

1 Introduction

Let $0 < n = 2k + 1 \in \mathbf{Z}$. We consider the complete graph K_n as an undirected Cayley graph Γ_n of the cyclic group \mathbf{Z}_n . The relations among totally multicolored copies of K_3 and K_4 in the Γ_n 's give rise to two families of structure graphs, namely: a family \mathcal{G}_0 of connected graphs G of largest degree $\Delta = 3$ with diameter asymptotically of order $|V(G)|^{1/2}$, [2], and a family \mathcal{G}_1 of connected graphs G of largest degree $\Delta = 6$ with diameter asymptotically of order $|V(G)|^{1/3}$, see Proposition 23 below, reached via the development in Sections 2-8. We will see that the graphs of \mathcal{G}_1 contain subgraphs modeled on the $(3, 6, 3, 6)$ -semiregular tessellation \mathcal{D} and on the $\{6, 3\}$ -regular hexagonal tessellation \mathcal{H} , [3]. Finally, we will prove the following pair of theorems.

Theorem 1 *The subset V_6 of vertices of degree 6 in a typical graph G of \mathcal{G}_1 is asymptotically of order $|V(G)|$. Each $v \in V_6$ is incident to three triangles T_0, T_1, T_2 of G with pairwise intersection $\{v\}$. Moreover, for each pair $\{i, j\} \subset \{0, 1, 2\}$ with $i \neq j$, the pair $\{T_i, T_j\}$ determines two different planar subgraphs $D_{i,j}^0$ and $D_{i,j}^1$ of G . This yields six \mathcal{D} -modeled subgraphs $D_{i,j}^k$, where $i, j = 0, 1, 2$ and $k = 0, 1$. In addition, every $v \in V_6$ belongs to four subgraphs of G modeled on \mathcal{H} . In sum, each $v \in V_6$ is an intersecting vertex of ten different tessellated subgraphs of G : six (four) in which the degree of v is 4 (3).*

Theorem 2 *The statement of Theorem 1 holds for a subfamily \mathcal{G}'_1 of \mathcal{G}_1 in which the following properties hold as well: The subgraphs $D_{i,j}^0$ and $D_{i,j}^1$ in Theorem 1 are contained in*

subgraphs of \mathcal{D} delimited by 30° - 60° - 90° triangles R in the Euclidean plane. The total number of such R 's is asymptotically of order $|V(G)|^{1/3}$. Moreover, these R 's are all pairwise isomorphic, and each of them have asymptotically $|V(G)|^{2/3}$ vertices.

We prove Theorems 1 and 2 at the end of Section 8, after presenting properties of the complete graph on $\mathbf{N} = \{m \in \mathbf{Z} : m \geq 0\}$ considered as an undirected Cayley graph of \mathbf{Z} , in Sections 2 through 7. On the other hand, the families \mathcal{G}_0 and \mathcal{G}_1 above inspire the following conjecture, for which they provide (radical) confirmations.

Conjecture 3 *The graph diameter in a family of graphs with a common largest degree is asymptotically of the order of a given (radical, logarithmic, ...) function of the vertex number.*

Theorem 4 *A radical confirmation of Conjecture 3, for $\Delta = 6$, apart from the families \mathcal{G}_1 and \mathcal{G}'_1 of Theorems 1-2, is by means of the initial member of a collection of families of Cayley graphs of regular degree $2m \geq 6$ with diameter asymptotically of order $|V(G)|^{1/m}$, where $3 \leq m \in \mathbf{Z}$.*

Proof. For n sufficiently large, the undirected Cayley graph $\Lambda_3(n)$ of \mathbf{Z}_n having 0 adjacent either to $1, -1, n^{1/3}, n^{-1/3}, n^{2/3}$ and $n^{-2/3}$ or to their nearest integers if n is not a cube, is 6-regular and vertex transitive. If n is not a cube, the following argument is slightly different. Let $x, y \in \mathbf{Z}_n$. There is a path P_1 in $\Lambda_3(n)$ from x to z_1 , where $|y - z_1| \leq n^{2/3}$, with edge differences $\pm n^{2/3}$. The length of P_1 is at most $\frac{n}{n^{2/3}} = n^{1/3}$. There is a path P_2 in $\Lambda_3(n)$ from z_1 to z_2 , where $|z_2 - z_1| \leq n^{1/3}$, with edge differences $\pm n^{1/3}$. The length of P_2 is at most $\frac{n^{2/3}}{n^{1/3}} = n^{1/3}$. There is a path P_3 in $\Lambda_3(n)$ from z_2 to y with edge differences ± 1 . The length of P_3 is at most $n^{1/3}$. The concatenation $P = P_1 P_2 P_3$ from x to y has length at most $3n^{1/3}$. Hence, the diameter of $\Lambda_3(n)$ is at most $3n^{1/3}$. Note that the length of the shortest path from 0 to $n/2$ consists entirely of edge differences equal to $\pm n^{2/3}$, and consequently has $\frac{n}{2} n^{2/3} = \frac{n^{2/3}}{2}$ edges. Thus, the diameter of $\Lambda_3(n)$ lies between $\frac{n^{1/3}}{2}$ and $3n^{1/3}$.

The argument above can be modified by replacing the denominator 3 in the exponents of n by any integer $m > 3$, provided n keeps being sufficiently large. This leads to a confirmation of Conjecture 3 by means of a family of Cayley graphs $\Lambda_m(n)$ of \mathbf{Z}_n with diameter asymptotically of order $n^{1/m}$. This is obtained via paths P_i , ($i = 1, \dots, m$), of lengths at most $n^{1/m}$ and edge differences $\pm n^{(m-i)/m}$, whose orderly concatenation starts at x and ends at y , with inner concatenation vertices z_1, z_2, \dots, z_{m-1} such that $|y - z_1| \leq n^{(n-1)/n}$ and $|z_{i+1} - z_i| \leq n^{(n-i)/n}$, for $1 \leq i \leq m - 2$. \square

2 K_3 -types and K_3 -type graphs

Let $\Gamma_n = \Gamma(\mathbf{Z}_n, I_n)$ be the undirected Cayley graph of \mathbf{Z}_n with $I_n = \{1, 2, \dots, k\}$ as generating set. The elements x of I_n , referred to as the *colors* of Γ_n , are in 1-to-1 correspondence with the pairs $\{x, -x\} \subset \mathbf{Z}_n \setminus \{0\}$. This insures Γ_n as an edge-colored version of K_n with degree 2 in each color at each vertex. Thus, Γ_n can be considered as an undirected edge-colored K_n . A triangle in Γ_n has K_3 -type (a, b, c) if its edges have colors $a, b, c \in I_n$.

If no confusion arises, we suppress commas and parentheses. More generally, a K_3 -type $abc = acb = bac = bca = cab = cba$ of \mathbf{Z}_n is a 3-multiset $\{a, b, c\}$ of $I_n \cup \{0\}$ such that $a + b \in \{c, -c\} \in I_n$, where $a + b$ is taken mod n . (This 3-multiset can be viewed as a class of at most six 3-tuples of colors of $I_n \cup \{0\}$, one of which is abc).

Example: The K_3 -types $\{a, b, c\}$ of \mathbf{Z}_7 with $\gcd(a, b, c) = 1$ are $\{0, 1, 1\}$, $\{1, 1, 2\}$, $\{1, 2, 3\}$, $\{1, 3, -(1+3) = 3\}$ and $\{2, 3, -(2+3) = 2\}$.

Let G_n be the graph whose vertices are the K_3 -types of \mathbf{Z}_n and such that any two such vertices, say v and v' , are adjacent by means of an edge ϵ if and only if v and v' share exactly two objects (or possibly one object repeated twice), say a and b , in which case ϵ is determined by $\{v, v'\}$, or by $\{a, b\}$. We take $\{a, b\}$ (or ab , for short) as the *color* of ϵ , so that G_n becomes an edge-colored graph. In addition, we may assume that G_n does not have multiple edges and we incorporate this assumption into the definition of G_n .

A complete subgraph of Γ_n is *totally multicolored* (or *TMC*) if its edges have different colors. In Example 5, only 123 is TMC. In Γ_n each TMC triangle t and edge ϵ of t determine exactly one TMC triangle $t' \neq t$ with the same colors of t and sharing ϵ with t .

For $n = 2k + 1 \geq 7$, let $G'_n \subseteq G_n$ be the subgraph of G_n induced by the TMC K_3 -types of \mathbf{Z}_n . Let $G_{n,3}$ be the component of G'_n containing the K_3 -type 123. Then all the other components of G'_n are isomorphic to graphs of the form $G_{m,3}$ with $1 < m < n$ and $m|n$. Notice that the vertices of $G_{m,3}$ are 3-sets.

Consider $\mathbf{N} = \{m \in \mathbf{Z} : m \geq 0\}$ as a color set. A K_3 -type abc of \mathbf{Z} , simply called a K_3 -type, is a 3-multiset $\{a, b, c\}$ of \mathbf{N} such that the sum of the two least colors equals the greatest one. Let $G_{\infty,3}$ be the graph whose vertices are the K_3 -types abc with $\gcd(a, b, c) = 1$ and whose edges are as defined above for G_n .

Given $m, m', n \in \mathbf{N}$ with $m' \in I_n$, we say that $m' \equiv m \pmod n$ if and only if for $m'' \equiv m \pmod n$ with $0 \leq m'' < n$: (1) if $m'' > n/2$, then $m' = n - m''$; (2) otherwise, $m' = m''$. Such m' is said to be the *reduction of $m \pmod n$* .

It was shown in [2], Proposition 2.16, that for odd $n \geq 7$, $G_{n,3}$ can be obtained, from a connected subgraph F of $G_{\infty,3}$ containing 011, 112, 123 and the other K_3 -types with colors $\leq n$, by reducing MOD n all the colors of K_3 -types of F .

Let $\phi(n)$ be the value of Euler's totient function at a positive integer n . It was shown in Theorem 2.17 of [2] that $|V(G_{n,3})| = O(n\phi(n))$ and subsequently, in Theorems 2.20 and 2.21, that the diameter of $G_{n,3}$ is both $\Omega(n)$ and $O(|V(G_{n,3})|^{1/2})$. The family \mathcal{G}_0 in the Introduction is formed by these graphs $G_{n,3}$.

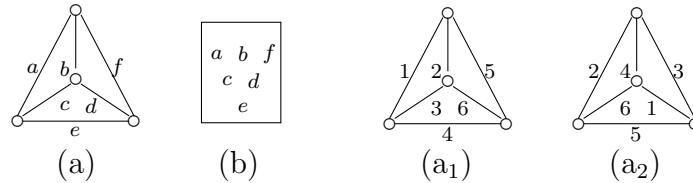


Figure 1: Representing a generic K_4 -type $abcdef$ and its cases MOD 13

3 K_4 -types and K_4 -type graphs

A K_4 -type of \mathbf{Z}_n , (\mathbf{Z}), is a maximal class of 6-tuples $abcdef$ of colors of I_n , (\mathbf{N}), such that abc , cde , $ae f$ and bdf are K_3 -types of \mathbf{Z}_n , (\mathbf{Z}). Note that such a maximal class has at most 24 6-tuples. A 6-tuple in a K_4 -type t is called a *card* of t . If no confusion arises, we represent a K_4 -type by any of its cards. A K_4 -type is *TMC* if its six colors are pairwise different. The card $abcdef$ is represented in two possible ways, (see left half of Figure 1): **(a)** either as a copy of K_4 each of whose edges bears a color; **(b)** or, more succinctly, by keeping only the locations of the color denominations in (a) inside a prototypical rectangular frame.

The colors in Figure 1(a) split into three different pairs of opposite colors: $\{a, d\}$, $\{b, e\}$, $\{c, f\}$, (opposite in the sense that each pair is held by a corresponding pair of edges of K_4 with no vertices in common, the remaining edges forming a 4-cycle).

Any 6-multiset of \mathbf{N} determines *at most* one K_4 -type (of \mathbf{Z}). This is not true for \mathbf{Z}_n in place of \mathbf{Z} : the two TMC K_4 -types 123645 and 246153 of \mathbf{Z}_{13} , represented in Figures 1(a₁) and 1(a₂), respectively, are different but have the same underlying (multi)set.

Given $n = 2k + 1 \geq 13$, let $G'_{n,4}$ be the graph whose vertices are the TMC K_4 -types $abcdef$ of \mathbf{Z}_n with $\gcd(a, b, c, d, e, f, n) = 1$ and such that any two such vertices, say t and t' , are adjacent by means of an edge ϵ if and only if t and t' , looked upon as K_4 -types, share exactly two K_3 -types v and v' . In this case, v and v' share exactly one color a of I_n . We take a as the color of ϵ , and this makes $G'_{n,4}$ into an edge-colored graph.

Figure 2 may be interpreted as a neighborhood N of the K_4 -type 123745 in a supergraph $G_{n,4}$ of $G'_{n,4}$ (defined in Theorem 18, below), where $n > 15$ is odd; (notice that the two lowermost-rightmost K_4 -types in Figure 2 are not TMC). Any edge ϵ with endvertices t, t' in such a neighborhood N is determined by the two K_3 -types v, v' that t and t' have in common. Motivated by this, we define subsequently a graph whose vertices are K_4 -types with adjacency of any two vertices if and only if they share exactly two K_3 -types.

Let $G''_{\infty,4}$ be the simple graph whose vertices are the K_4 -types $abcdef$ with $a \neq d$, $b \neq e$ and $c \neq f$ unless $abcdef = 011011$ and satisfying $\gcd(a, b, c, d, e, f) = 1$, with two vertices u and v determining an edge ϵ if and only if they share exactly two K_3 -types.

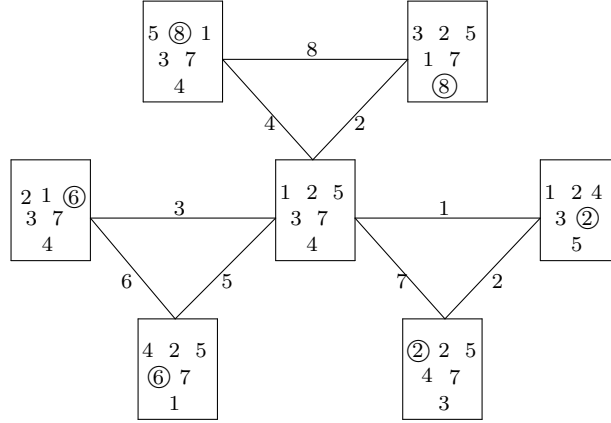


Figure 2: A neighborhood of 123745 in $G''_{\infty,4}$

Figure 2 can be used to illustrate Theorem 5, below. An edge ϵ joining two vertices t and t' of $G''_{\infty,4}$ with respective cards r and r' determines a K_3 -type s common to t and t' and *equally located* in r and r' , in the sense that the component colors of s occupy the same positions in r and r' , (just as the K_3 -type $s = 123$ is not only common but, moreover, equally located in both the central card in Figure 2 and the one horizontally located at its right, where s occupies the three uppermost-leftmost locations in r and r'). For ease of comprehension, the locations g_r of the colors in the cards r' of the statement of Theorem 5 obtained from the central card r in the center of Figure 2 are shown encircled.

Theorem 5 *Let $t \in V(G''_{\infty,4})$ have a card r with color g at location g_r and color g' at the location g'_r opposite to g_r . Then t has a neighbor t' with card r' differing from r just in: (a) the color at g_r and (b) a permutation of the colors at the two locations $\neq g'_r$ in just one of the two K_3 -types common to r and r' that contain the color at g_r .*

Proof. t' is determined from t as follows. Let s, s' be the two K_3 -types not containing g_r in r . Then s and s' contain g'_r . We can assume s' has its colors equally located in r and r' . Let i, j be the colors of r at the two locations $i_r, j_r \neq g'_r$ of s . Thus $s = ijg'$. The other two K_3 -types in t apart from s and s' are of the form gij' and ggi' , with $s' = i'j'k$. We interpret r' as having the colors i, j exchanged, with respect to r . That is to say $(i_{r'}, j_{r'}) = (j_r, i_r)$. Let $\nu(a, b) = \{|a - b|\} \cup \{a + b\}$, for each pair of integers $a, b \geq 0$. There is at least a color $h \in \nu(i, j) \cap \nu(i', j') \neq \emptyset$ that yields r' when located at g_r (which should be called $h_{r'}$ in r') so that r' is formed by the K_3 -types $s = ijg'$, $s' = i'j'g'$, hii' and hjj' . Moreover, r' does not depend on the selected card r of t . In fact $h = h(r, g_r)$ depends only on r and g_r . If $r = 011011$ and $g = 0$ then h equals either 0, yielding $t' = t$, not a distinct neighbor of t in $G''_{\infty,4}$ so we discard it, or 2, yielding a neighbor t' of t . Otherwise, since no other vertex of $G''_{\infty,4}$ is of the form $abcabc \neq 011011$, then $|\nu(i, j) \cap \nu(i', j')| = 1$, even if $(r, g) = (011011, 1)$. Thus, if either $r \neq 011011$ or $(r, g) = (011011, 1)$, then h is unique. \square

Example: In the following special cases, g assumes subsequently colors f, a and d in a K_4 -type t of card $r = abcdef$: **(A)** applying Theorem 5 to $(r, g) = (112354, 4)$, (so $g = f$), yields $t' = t$, where $g_r = f_r = 4_r$, because exchanging $d_r = 1_r$ and $e_r = 1_r$ does not produce changes from r ; **(B)** applying Theorem 5 to $(r, g) = (011011, 0)$, (so $g = a, d$), yields for $h = 2$ neighbors t', t'' with respective cards $r' = 211011$ and $r'' = 011211$, where $g_r = a_r, d_r$, respectively, but observe that $t' = t''$.

4 Canonical triangles and connectedness

Let $G_{\infty,4}$ be the supergraph of $G''_{\infty,4}$ obtained by adding to the vertices of $G''_{\infty,4} \setminus \{011011\}$ the loops offered by the method of vertex adjacency suggested in Figure 2 and Theorem 5, taking each loop with multiplicity 1. Then, any edge (or loop) joining vertices t and t' in $G_{\infty,4}$ has the pair (s, s') in the proof of Theorem 5 as its *strong color* and the only color g' in s and s' that remains at the location $g'_r = g'_{r'}$, both in r and r' , as its *weak color*. Let $G'_{\infty,4}$ be the graph obtained from $G_{\infty,4}$ by restriction to the TMC K_4 -types.

Corollary 6 *The graphs $G'_{\infty,4}$ and $G'_{n,4}$ are edge-disjoint unions of triangles, at most three such triangles incident to each vertex.*

Proof. Applying Theorem 5 to the colors g, g' of a pair of opposite edges of a vertex t of $G_{\infty,4}$ looked upon as a K_4 -type with card r yields $h(r, g) = h(r, g')$. This determines two respective neighboring cards r' and r'' of r , which represent neighbors t' and t'' of t , respectively. The two K_3 -types that r' and r'' share and those two that r and r' , (r and r''), share constitute the four K_3 -types of r' , (r''). Finally, each $G'_{n,4}$ can be obtained from $G'_{\infty,4}$ via reduction MOD n . \square

The triangles in Corollary 6 are called *canonical triangles*. When two or three K_4 -types in a CT $T = \{t, t', t''\}$ obtained as in Theorem 5 coincide. $t = t' \neq t''$ or $t = t'' \neq t'$ or $t \neq t' = t''$ or $t = t' = t''$.

Example: (A) If t has $r = abcdef$ with $a, b > 0$, $c = a + b$, $d = a$, $e = b$, $f = |a - b|$ and $(g_r, g'_r) \in \{(a_r, d_r), (b_r, e_r)\}$, then $t' = t''$. This yields two degenerate CT's, with vertices of the form $t, t', t'' = t'$, where $tt' = tt''$ and $t't''$ is a loop of $G_{\infty,4}$. (B) Theorem 5 applied to $t = 000111$ yields three degenerate CT's, each representable by just two vertices, namely t (twice) and $t' = 011011$, a nonloop edge tt' and a loop at t ; these three CT's coincide, since edges are assumed to have multiplicity 1. (C) Theorem 5 applied to $t = 132112$ yields three CT's incident to t , one of which, obtained by making value changes in both cases of color $g = 2$ at opposite locations in t , has its three vertices equal to t , so this CT reduces to a looped vertex in $G_{\infty,4}$. The other two CT's incident to t are $\{t, 202111, 132201\}$ and $\{t, 431122, 132421\}$.

Theorem 7 $G_{\infty,4}$ is connected.

Proof. For any $t = abcdef$ and $t' = abcydx$, there exists a 2-path in $G_{\infty,4}$ from t to t' with middle vertex card $abcfxd$ and edge strong colors $\{abc, bdf\}$ and $\{abc, adx\}$. Let cde and cxy be K_3 -types of \mathbf{Z} with $\gcd(c, d, e) = \gcd(c, x, y)$. Then there exists a path in $G_{\infty,4}$ whose ends have cards of the form $abcdef$ and $abxyz$. This uses the fact that if $\gcd(c, d, e) = \gcd(c, x, y)$, then there is a path in $G_{\infty,3}$ from cde to cxy , [2]. Thus, if $abcdef \in V(G_{\infty,4})$, then there exist: (a) a path in $G_{\infty,4}$ from 110110 to $110aa(a+1)$; (b) a path in $G_{\infty,4}$ from $110aa(a+1)$ to $aa0bbc$; (c) a path in $G_{\infty,4}$ from $aa0bbc$ to $abcdef$. Hence, every vertex of $G_{\infty,4}$ can be connected to 110110 . \square

5 A planar-subgraph generating algorithm

Theorem 8 *The CT's of $G_{\infty,4}$ are in 1-1 correspondence with the family of multisets $abcd$ of colors of \mathbf{N} such that: (a) $\nu(a, b) \cap \nu(c, d) \neq \emptyset$ (or $\nu(a, c) \cap \nu(b, d) \neq \emptyset$ or $\nu(a, d) \cap \nu(b, c) \neq \emptyset$). (b) $\gcd(a, b, c, d) = 1$, so at least one of a, b, c, d is nonzero.*

Proof. From Theorem 5 and Corollary 6, each CT of $G_{\infty,4}$ has its vertices as K_4 -types sharing exactly four colors as in the statement. \square

Example: In Figure 2, the upper, (lower-left, lower-right), CT has its vertices sharing the quadrangle 1357, (1247, 2345).

From now on, each CT is denoted by its associated multiset as in Theorem 8. Given a TMC K_4 -type $t = abcdef$, the CT's incident to t are obtained by deleting from t each one of the three pairs ad , be and cf , yielding respectively $bcef$, $acdf$ and $abde$.

Consider the union $C \cup D$ of any two CT's C and D of $G_{\infty,4}$ having a vertex $abcdef$ in common. To fix ideas let $C = acdf$ and $D = abde$. We set $C \cup D$ as a plane graph $B(t, a, d)$, as follows. C and D are represented by congruent equilateral triangles \overline{C} and \overline{D} in the Euclidean plane with color a indicating their centers and with the other colors of C and D indicating internally (in \overline{C} and \overline{D}) the vertices of \overline{C} and \overline{D} , respectively, where d is the color indicating t in both \overline{C} and \overline{D} . The sides of \overline{C} and \overline{D} incident to t are drawn on two straight lines at external angles of 120° . We color each edge of \overline{C} , (\overline{D}), with the weak color of the corresponding edge of C , (D). The weak color of each edge ϵ of \overline{C} forms: **(a)** a K_3 -type $s(\epsilon)$ together with the colors indicating the endvertices of ϵ in \overline{C} ; **(b)** another K_3 -type $s'(\epsilon)$, together with the central color of \overline{C} and the color indicating the vertex opposite to ϵ in \overline{C} . Notice that $\{s(\epsilon), s'(\epsilon)\}$ is the strong color of the image of ϵ in $G_{\infty,4}$.

Let ϵ_C and ϵ_D be edges of \overline{C} and \overline{D} , respectively, meeting at an angle of 120° at vertex t . Then the color d indicating t in both \overline{C} and \overline{D} forms with the colors of ϵ_C and ϵ_D the K_3 -type $s(\epsilon_C) = s(\epsilon_D)$.

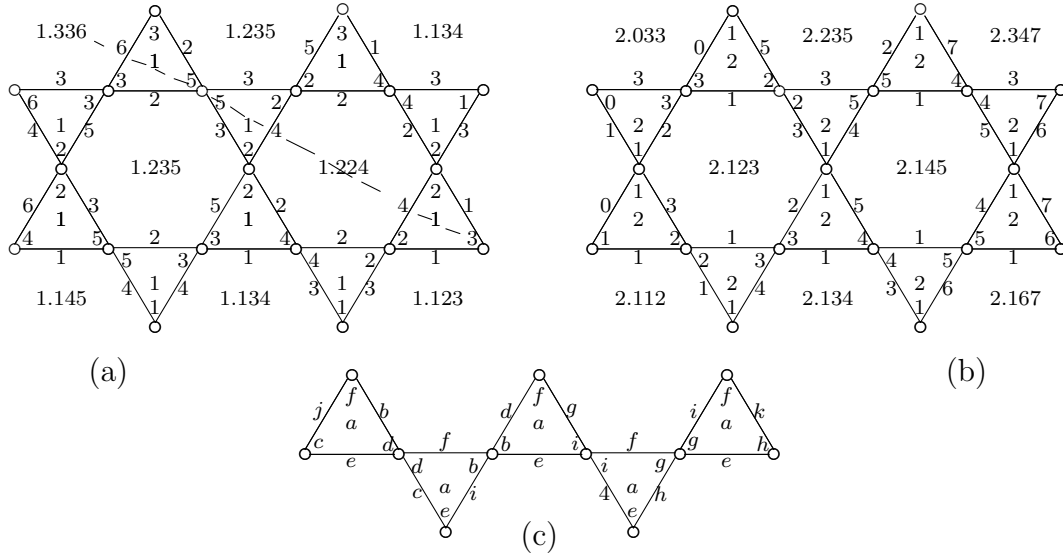


Figure 3: Unfolded coverings of subgraphs of $G_{\infty,4}$

Instances (a) and (b) of Figure 3 are illustrations for Proposition 9 adequate in order to make grow a planar subgraph of $G_{\infty,4}$ starting out from $B(t, a, d)$.

Proposition 9 (1) Given a CT $C = afgh$, let a be the central color of \overline{C} and let color f indicate a vertex u of \overline{C} internally in \overline{C} . Then there exists a color i such that **(a)** $\nu(a, h) \cap \nu(f, g) = \{i\}$; **(b)** the edges $\epsilon = uu'$ in \overline{C} such that u' is colored with g or h inside \overline{C} have color $\gamma(\epsilon) = i$. **(2)** Let ℓ be the line passing through u parallel to the unique edge of $\overline{C} \setminus u$.

Then each pair (u, C) determines at most one other CT $D \neq C$ sharing u with C , such that $\overline{D} = \rho_\ell(\overline{C})$, where ρ_ℓ is reflection of the plane on ℓ , and having (a) a as central color; (b) the color f of u inside \overline{C} as color of u inside \overline{D} ; (c) for each one of the two edges $\epsilon = uu'$ of \overline{C} : (i) $\gamma(\epsilon)$ as the color of $\rho_\ell(u')$ in \overline{D} and (ii) the color of u' in \overline{C} as the color of $\rho_\ell(\epsilon)$. (3) The vertex u is the K_4 -type formed by the K_3 -types determined by each one of the two edges ϵ of \overline{D} incident to u and formed by: (a) a and the colors coloring ϵ and the vertex opposite to ϵ in \overline{D} ; (b) the colors coloring ϵ and the endvertices of ϵ in \overline{D} .

Proof. This follows by a series of applications of the previous ideas, via Theorem 5. \square

The union of two CT's C and D that share exactly one vertex v is said to be a *butterfly* and denoted CvD . In this case, v is said to be the *central vertex* of CvD . Note that the colors of v in \overline{C} and \overline{D} equal a fixed color d , which we call the *butterfly color* of CvD . For example, $B(t, a, d)$ above is a butterfly CtD with central color a and butterfly color d , say with $C = acdf$ and $D = abde$.

Corollary 10 *Let $t = abcdef$ be a TMC K_4 -type. Any sequence of applications of Proposition 9 starting at the two CT's in $B(t, a, d)$, followed by all posterior induction-step applications in all permissible direction instances, constitutes an algorithm that yields a planar graph $H' = H'(t, a) = H'(t, a, d)$, which is a covering of an edge-disjoint union $H = H(t, a) = H(t, a, d)$ of butterflies in $G_{\infty,4}$ having common central color a .*

Example: Both Figure 3(a) and 3(b) show parts of an H' as in Corollary 10. In fact H' is the $(3, 6, 3, 6)$ -semiregular tessellation. We will see that if such an H' is not a subgraph of $G_{\infty,4}$, then it can be folded along at most two *symmetry axes* (SA's) to yield H . The dotted line in Figure 3(a) represents such an SA. In particular, colors will coincide by reflection in an SA. H will be seen to be a subgraph of H' spanning a connected region of the plane delimited by SA's. Edges crossing an SA at 90° will yield loops of H and each CT in H' will be incident to three hexagons.

Proposition 11 *Given a vertex t of $H'(t, a, d)$, the three CT's incident to t according to Theorem 5 are: (a) the two CT's incident to t in $H'(t, a, d)$ and (b) the CT formed by the colors of the four edges of the two CT's in item (a) which are incident to t .*

6 Canonical hexagons

Passing through each one of its vertices, $H'(t, a, d)$ has two 6-cycles given by regular hexagons in the plane whenever the CT's of $H'(t, a, d)$ are represented by equilateral triangles as in Section 5, with their sides representing the edges of those CT's. This is the specific situation of Proposition 12, below. If q is any of these 6-cycles, then its edges are colored with the component colors of a K_3 -type s . In that case, we denote $q = a.s$, where a is the central color of the six CT's adjacent to q .

Proposition 12 *Let $bdf = s$ and $cde = s'$ be K_3 -types. Then $t = abcdef$ is contained in 6-cycles $q = a.s$ and $q' = a.s'$ of $H'(t, a, d)$. The edge-color sets of q and q' are respectively*

$\{b, d, f\}$ and $\{c, d, e\}$, each of their elements coloring opposite edges. Moreover, the color that indicates t internally in its incident CT's in $H'(t, a, d)$ and those that indicate the two edges in $q, (q')$, incident to t conform $s, (s')$. Furthermore, d is the color that colors t in its incident CT's in $H'(t, a, d)$, as well as the two parallel edges of $a.bdf, (a.cde)$, incident neither to t nor to its corresponding opposite vertex.

Proof. We will see that there exists a 6-cycle $(t^0, t^1, t^2, t^3, t^4, t^5)$ of vertices of $H'(t, a, d)$ passing through $t = t^0$ and determined by means of the following algorithm, that yields t^i when t^{i-1} is given, for $i = 1, 2, 3, 4, 5$, (and returns to $t^0 = t^i$ from $t^5 = t^{i-1}$, if $i = 6 \equiv 0$ with indices taken mod 6): **(a)** declare the card r^i of the K_4 -type t^i to have color a (as in Figure 1(b)) fixed in the location a_{r^0} (so that $a_{r^i} = a_{r^0}$) through the entire algorithm; **(b)** denote locations $b_{r^i} = b_{r^0}$, $c_{r^i} = c_{r^0}$ and $e_{r^i} = e_{r^0}$, regardless of changes in their color values from their initial values, namely b, c and e , respectively, through the running of the algorithm; **(c)** define color $h^i = b, (h^i = f)$, if i is even, (odd); **(d)** establish an exchange of colors, indicated via a new denomination of locations at the i -th level: $d_{r^i} = h_{r^{i-1}}^{i-1}$ and $h_{r^i}^i = d_{r^{i-1}}$; **(e)** the color $e_{r^i}, (c_{r^i})$, if i is even, (odd), takes the only value from $\nu(a_{r^i}, f_{r^i}) \cap \nu(c_{r^i}, d_{r^i})$, ($\nu(a_{r^i}, b_{r^i}) \cap \nu(d_{r^i}, e_{r^i})$); this determines a well-defined r^i . We just gave one of the location instances for the determination of a 6-cycle as claimed, the other ones being essentially equivalent to this one. The rest of the statement follows. \square

Example: A 6-cycle generated textually as in the proof of Proposition 12 and starting at $t^0 = 123745$ is $a.s = 1.257 = (123745, 123587, 156287, 156712, 176512, 176245)$. Its accompanying coplanar 6-cycle $a.s'$ is $1.347 = (123745, 187345, 187434, 134734, 134376, 123476)$. An essentially equivalent 6-cycle to these and sharing its first two vertices with $a.s'$ as just given, is $7.145 = (123745, 583741, 48c751, 1bc754, 5b6714, 426715)$, where lowercase hexadecimal notation is used, and its accompanying coplanar 6-cycle is $7.123 =$

$$(123745, 321785, 23178a, 13279a, 312796, 213746),$$

sharing its first two vertices with $a.s$.

Given K_3 -types bcd and $bc'd'$ with $b < c < d$ and $b < c' < d'$, define $bcd < bc'd'$ if and only if $c + d < c' + d'$. A graph $H' = H'(t, a, d)$ as in Corollary 10 is said to be a T-subgraph and denoted $a(s)$, where s is the smallest K_3 -type $\neq 000$ coloring a 6-cycle of H' under ' $<$ ', while $H = H(t, a, d)$ is denoted $a[s]$. Hexagons $a.s$ of an $H'(t, s, d)$ and their images in $H(t, a, d)$ are called *canonical hexagons* (or CH's).

Proposition 13 *Let $H' = H'(t, a, d)$, where $t = abcdef$ is common to $C = acdf$ and $D = abde$, with $\overline{C} \cup \overline{D} \subset H'(t, a, d)$ and d indicating t internally in both \overline{C} and \overline{D} . Then, the T-subgraph $H'' = H'(t, d, a)$ has t common to a flipped copy $\overline{\overline{D}}$ of \overline{D} and a direct copy $\overline{\overline{C}}$ of \overline{C} . As a result, $d.caf$ and $d.bae$ are the colors of the CT's incident to t in H'' . Moreover, $H'' = H'$ if and only if $f = c$ and $e = b$.*

Proof. $H'' = H'(t, d, a)$ is given as follows: **(1)** represent H'' as a temporarily uncolored T-subgraph and set t as one of its vertices; **(2)** represent $\overline{\overline{C}}$ and $\overline{\overline{D}}$ in H'' as the respective CT's \overline{C} and \overline{D} of H' with common vertex t but set the locations of a and d in $\overline{\overline{C}}$ and $\overline{\overline{D}}$,

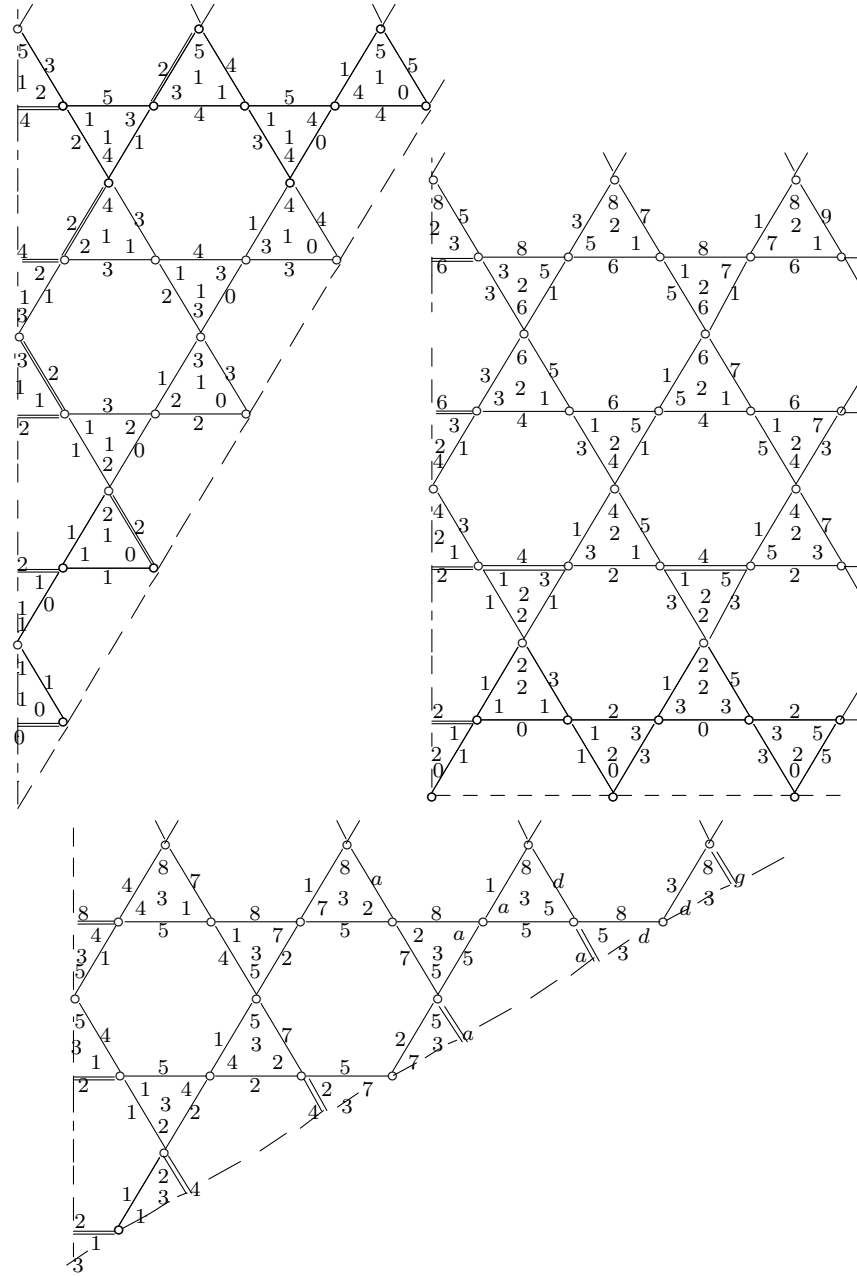


Figure 4: Charts for 1[011], 2[011] and 3[112]

instead, as those of d and a in \overline{C} and \overline{D} , respectively; **(3)** the vertex colors c and f in \overline{C} are exchanged with respect to their locations in \overline{C} while the two vertex colors b and e in \overline{D} are left as in \overline{D} . The remaining colors of H'' can be set uniquely by means of Proposition 9. If $H'' \neq H'$, then reflection with respect to the line perpendicular to the line ℓ of Proposition 9 through t takes each edge color of \overline{D} in H'' to its location in \overline{D} , while the edge colors of \overline{C} remain as in \overline{C} . The statement follows immediately, as illustrated in Figure 3, where (b), at right, represents part of the T-subgraph H'' corresponding to the T-subgraph H' , partly represented itself in (a), with $t = 235142$ at the center in both representations. \square

7 Canonical charts

Partial pictures of some graphs $a[s] = a[bcd]$ of $G_{\infty,4}$ are given in Figure 4, with notation established just before Proposition 13 and edges in single, (double), trace representing nonloop edges, (loops). In fact, these partial pictures yield subgraphs of the corresponding graphs $a(s) = a(bcd)$, in which both single- and double-trace edges now represent nonloop edges. Concretely, Figure 4 upper-left, (upper-right), shows a plane region delimited by two dotted lines ℓ and ℓ' that form an internal angle of 30° , (90°), and determine a partial representation of $H'(s,1) = 1(011)$, ($H'(s,2) = 2(011)$), where $s = 110001$, ($s = 211011$). This representation can be identified with $H(s,1) = 1[011]$, ($H(s,2) = 2[011]$), by interpreting each double-trace edge interrupted perpendicularly by some dotted line ℓ as bouncing, or reflecting, at 90° on ℓ , or equivalently, interpreting it as a loop. Moreover, $H'(s,1)$, ($H'(s,2)$), is obtained by unfolding $H(s,1)$, ($H(s,2)$), along the symmetry axes in the finite sequence $\ell_0 = \ell$, $\ell_1 = \ell'$, \dots , $\ell_i = (\text{reflected line of } \ell_{i-2} \text{ on } \ell_{i-1})$, for $i = 2, \dots, k-1$, where additionally $\ell_{k-1} = (\text{reflected line of } \ell_1 \text{ on } \ell_0)$, with $k = 360/30 = 12$, ($k = 360/90 = 4$).

The extensions of these partial pictures to the plane will be referred to as *charts*. Then, the two charts above are the only ones that are of the form $H'(t,a)$, with $a = 1, 2$. However, no other value of a produces just one chart. For example, there are two charts $H'(s,3)$, one of which is $3(112)$, with $3[112]$ partially shown in the bottom of Figure 4, where two straight lines ℓ_0 and ℓ_1 at an angle of 60° delimit its representation, and with finite sequence ℓ_0, ℓ_1, \dots as above having length $k = 360/60 = 6$. The remaining $H'(s,3)$ is $3(011)$, with $3[011]$ having exactly one SA, delimiting its necessarily resulting semi-plane representation. As a increases its value, the first chart H not having an SA is $H = 6(123) = 6[123]$. The following proposition specifies how the unfolding of a graph $a(bcd)$ onto its corresponding $a[bcd]$ takes place. Proofs in this section are omitted, as they follow by inspection.

Proposition 14 *If $H(t,a) \neq H'(t,a)$, then $H(t,a)$ is obtained by means of folds of $H'(t,a)$ along SA's of two types: **(1)** SA's dividing all CH's of the form $a.0cc$ in symmetric halves through vertices colored with 0 in CT's of the form $a0cd$, i.e. through all vertices of the forms $0bbcca$ and $0ccdda$; **(2)** SA's dividing all CH's of the form $a.bbc$ in symmetric halves, and passing at 90° through the midpoints of their edges colored with c , (double-trace edges that yield loops), and through the vertices opposite to them in corresponding CT's. Here, only the CT of the form $3(123)$ has two such SA's.*

In a chart H' , a double-trace edge halved perpendicularly in its middle point by some SA yields a *half-edge* of H , and a CT that contains a half-edge yields a $\frac{1}{2}$ -CT of H . Degenerate CT 1113, shown in the lower-left corner of the chart 3[112] in Figure 4, has its center as the intersection of two SA's, (three in 3(112)), and constitutes the only $\frac{1}{6}$ -CT of any chart of $G_{\infty,4}$. See also Example 9(C), where the CT's in their shown order are 1113, 1122 and 1123, the first two present in 3[112].

Corollary 15 (1) *A maximal connected region of an $H'(t, a)$ delimited by SA's but with its interior not intersecting any other SA yields a chart of $H(t, a)$.* (2) *Charts $a(bcd)$ and $a[bcd]$ exist, for $b \leq c \leq d$, if and only if $c + d \leq a$.* (3) *Every loop of $G_{\infty,4}$ not in CT's 0011, 1111, 0112, 1113 appears as a half-edge in two different charts and as a double-trace edge in a different one. The CT that contains such a loop: (a) is of the form $aabc$, where a, b, c are pairwise different and $(2a, b, c)$ is a K_3 -type; (b) appears as a $\frac{1}{2}$ -CT obtained by halving a degenerate CT as in Example 9(A) by means of an SA in $b[112]$ or $c[112]$, and as a 3-cycle in $a[011]$.*

The proofs of the facts stated below arise from chart inspection and are omitted. Two edges in a butterfly $B(t, a, d)$ are said to be *opposite* if each one of them does not have t as an endvertex. Note that each butterfly has just one pair of opposite edges.

Proposition 16 *Any infinite path of $H' = H'(t, a) = a(bcd)$ contained in a line has successive edge colors with alternating values f and g either differing in or adding up to a , the latter case occurring exactly if $f, g \leq a$. Denoting such a path by $L(f, g, a)$, we have: (1) $f = g$ whenever $f = a/2 \in \mathbf{Z}$ or $g = a/2 \in \mathbf{Z}$; in this case, $d = a/2$ if $d \geq b, c$; (2) the edges colored $2a$ in $L(a, 2a, a)$ are double-trace. If two such paths are parallel and contiguous in H' then they are of the form $L(f, g, a)$ and $L(h, f, a)$, with $|g - h| = 2a$ or $g + h = 2a$, the latter case occurring exactly if $g, h \leq 2a$. Here, g, h are the opposite edge colors of the butterflies taking place between $L(f, g, a)$ and $L(h, f, a)$. The edges of $L(f, g, a)$ and $L(h, f, a)$ colored with f are divided into pairs of opposite edges of the CH's lying between $L(f, g, a)$ and $L(h, f, a)$.*

Corollary 17 *Given a vertex v of $H(t, a)$, let f, g, h, i be the colors of the edges incident to a covering vertex of v in $H'(t, a)$. If a is odd or if v is not in an $L(a/2, a/2, a)$ then there is exactly one other vertex u of H such that the edges incident to any covering vertex of u in H' have colors f, g, h, i . In this case u and v belong to $s = fghi$ and the edge uv has color a . We may assume that v is shared in $H(t, a)$ by $a.fgj$ and $a.hij$, so that the edge of s having v as an endvertex but not u is colored with j , so j colors v in s .*

8 K_4 -types of \mathbf{Z}_n

Proposition 18 *Let $0 < n = 2k + 1 \in \mathbf{Z}$. There is a colored supergraph $G_{n,4}$ of the graph $G'_{n,4}$ introduced in Section 3 and a well-defined transformation Φ_n from $G_{\infty,4}$ onto $G_{n,4}$ that operates by replacing all colors of \mathbf{N} intervening in vertices, edges, CT's and CH's of $G_{\infty,4}$ by their image colors under reduction MOD n in the sense that all vertices, respectively edges, having a common image color can be identified.*

Proof. Let A be the subset of vertices of the graph $G_{\infty,4}$ introduced in Section 4 whose colors have constituents $\leq k$ and let B be the set of neighbors of vertices of A in $G_{\infty,4}$. Let F be the graph induced by $A \cup B$ in $G_{\infty,4}$. By reducing MOD n all the colors which are constituents of colors of F , the resulting color identifications in F yield $G_{n,4}$. Of course, the reductions MOD n happen solely for the vertices of B . Once these vertices are reduced MOD n , they have the same colors as some vertices of A , so they must be identified correspondingly, and the edges from A to B are then transformed into edges joining vertices of A which were not originally induced by Ag in $G_{\infty,4}$. Now, Φ_n is defined by replacing the colors of the objects in $G_{\infty,4}$, (vertices, edges, CT's and CH's), by their reductions MOD n , which yields the corresponding objects in $G_{n,4}$. \square

Corollary 19 *The graph $G_{n,4}$ is an edge-disjoint union of CT's, possibly degenerate, at most three incident to each vertex.*

Theorem 20 *$G_{n,4}$ is connected, for any odd positive integer n .*

Proof. Apply Theorems 7 and 18 to the (continuous) map $\Phi_n : G_{\infty,4} \rightarrow G_{n,4}$. \square

Application of Φ_n to the charts of $G_{\infty,4}$ yields charts of $G_{n,4}$. The collection of charts of $G_{n,4}$, $(G_{\infty,4})$, whose CT centers are colored i , for each $i \in \{1, \dots, n/2\}$, is called an i -atlas.

Proposition 21 *Let $\rho_n : I_n \rightarrow \{\text{atlases of } G_{n,4}\}$ be the assignment given by $\rho_n(i) = i$ -atlas of $G_{n,4}$, for each $i \in I_n$. If $\gcd(n, i) = 1 < i < n/2$, then $\rho_n(i)$ is obtained from $\rho_n(1)$ by replacing each color a indicating a vertex, edge, CT or CH of $\rho_n(1)$ by the reduction MOD n of $a.i$. If n is prime, applying Φ_n to the i -atlases of $G_{\infty,4}$ yields $\lfloor n/2 \rfloor$ i -atlases of $G_{n,4}$, which are graph isomorphic.*

Proof. The given modular reduction identifies oppositely signed colors mod n . \square

Chart $\rho_{13}(1)$, depicted in Figure 5, is an example of the following proposition.

Proposition 22 *Chart $\rho_n(1)$ is representable inside a plane triangle $T(n, 1)$ whose sides are SA's of the subgraph $1[011] \subset G_{\infty,4}$, namely two SA's of type (2) and one of type (1), as in Proposition 14. The internal angle between the SA's of type (2) is 60° . The internal angles between each of these and the SA of type (1) are 30° and 90° , respectively. The angle of 30° has its vertex at the center v of the CH 1.000 and $\rho_n(1)$ is represented in a twelfth part of the total angle of 360° at v . The angle of 90° has its vertex at $0jj1jj$, where $j = (n - 1)/2$. There is only one maximal path $L_{n,1}$ of $\rho_n(1)$ passing through $0jj1jj$ with its edges having color j and cutting the opposite side of $T(n, 1)$ at 90° on a double-trace edge. The angle of 60° has its vertex at the center of the CT $1hhh$, where $h = (n - 5)/2$.*

Proof. The statement follows by combining the images of the subgraphs $1[011]$, $2[011]$, $3[112]$ under the isomorphisms $\rho_n(1) \rightarrow \rho_n(i)$. \square

Proposition 23 *The diameter of $G_{n,4}$ is both $\Omega(n)$ and $O(|V(G_{n,4})|^{1/3})$, so it is asymptotically of order $|V(G_{n,4})|^{1/3}$.*

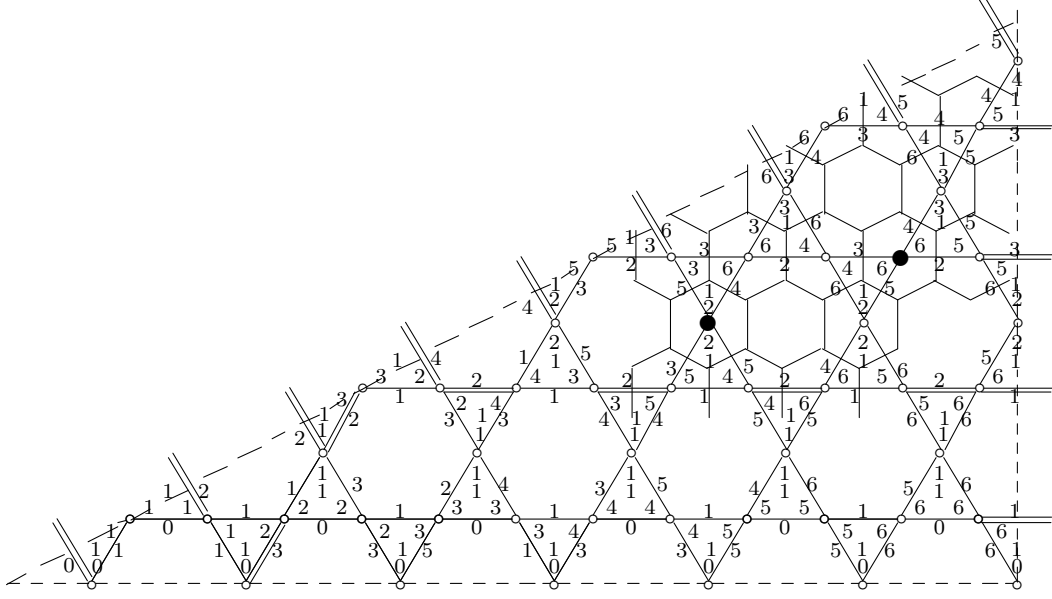


Figure 5: Superposition of drawings for $\sigma_n(1)$ and $\tau_n(1)$

Proof. Let us see that $|V(G_{n,3})| = O(n\phi(n))$, where $\phi(n)$ = Euler characteristic of n . Every $aa0$, where $\gcd(a, n) = 1$, belongs to $G_{n,3}$. Thus, there are $\lfloor \phi(n)/2 - 1 \rfloor$ paths whose ends are 011 and $0aa$, with $0 < a \leq \lfloor n/2 \rfloor$ and $\gcd(a, n) = 1$. But the distance from $0aa$ to 011 in $G_{n,3}$ is $\leq a$, yielding our claim. Second, let us see that $|V(G_{n,4})| = O(n^2\phi(n))$. If we fix a K_3 -type of $abcdef \in G_{n,4}$, say abc , then for each color $d \text{ MOD } n$ there are at most two different values for e , yielding a unique value for f . This way, there are at most $n\phi(n)(2\lfloor n/2 \rfloor)$ different K_4 -types MOD n , yielding our claim. Let us see now that the diameter of $G_{n,4}$ is $\Omega(n)$. A path of length $n+1$ between 110110 and $112(n-1)nn$ happens along the image of $L(1, 2, 2)$. Thus, the diameter of $G_{n,4}$ is $\Omega(n)$ and $O(|V(G_{n,4})|^{1/3})$ \square

A representation of the charts of $G'_{n,4}$ leading to the connectedness of $G'_{n,4}$ for n large is presented. Let $\sigma_n(1)$ be the restriction of $\rho_n(1)$ induced by the TMC vertices. We superpose the T-subgraph representation of $\sigma_n(1)$ with a $\{6, 3\}$ -regular hexagonal tessellation $\mathcal{H} = \tau_n(1)$, $[3]$, such that: **(a)** each edge ϵ of $\sigma_n(1)$ is traversed by an edge ϵ' of $\tau_n(1)$ at 90° at the common midpoint of ϵ and ϵ' ; **(b)** each CH of $\sigma_n(1)$ contains in its interior a regular hexagon of $\tau_n(1)$. Figure 5 contains a superposition of a representation of $\sigma_{13}(1)$, with the two TMC vertices distinguished, and the part of $\tau_{13}(1)$ used to represent $\sigma_{13}(1)$ in Figure 6.

In Figure 6, representing $\tau_n(1)$ for odd $n = 13, \dots, 25$, each TMC vertex of $\sigma_n(1)$ is given by an hexagon of $\tau_n(1)$ in which a positive integer is written. Each nonempty hexagon (or NH) representing a vertex of $\sigma_n(1)$ is the intersection of two NH sequences in $\tau_n(1)$. There are three directions of parallelism for appearing NH sequences: one horizontal and the other two at angles of $\pm 60^\circ$. Each such sequence is headed on the boundary of $\tau_n(1)$ by a partially-drawn thick-trace hexagon containing a pair of integers. Assume the integer inside an hexagon ζ of $\tau_n(1)$ is i and the integer pair heading its two NH sequences are (p, q) and (r, s) . Then the K_3 -types composing ζ are: $1pq$, $1rs$ and either ipr and iqs or ips

and *igr*. An hexagon contains a bullet \bullet instead of an integer if it represents a non-TMC K_4 -type. Each empty hexagon stands for a corresponding CH. It follows that each $\sigma_n(1)$ has at least two isolated vertices, represented in $\tau_n(1)$ by: **(1)** the hexagon containing 2, at the lower-left corner of $\tau_n(1)$, (K_4 -type 134265); **(2)** the hexagon containing $\lfloor n/2 \rfloor$, at the lower-right corner of $\tau_n(1)$, (K_4 -type $123k(k-2)(k-1)$, where $n = 2k+1$). If $n \neq 0 \pmod 3$ then these are the only two isolated vertices of $\sigma_n(1)$. Otherwise, there is exactly one more isolated vertex in $\sigma_n(1)$, determined by the hexagon containing $n/3$, at the upper-right corner of $\tau_n(1)$, (K_4 -type $1(k-2)(k-1)k(k+1)(k+2)$), as shown in Figure 6 for $n = 15, 21$,

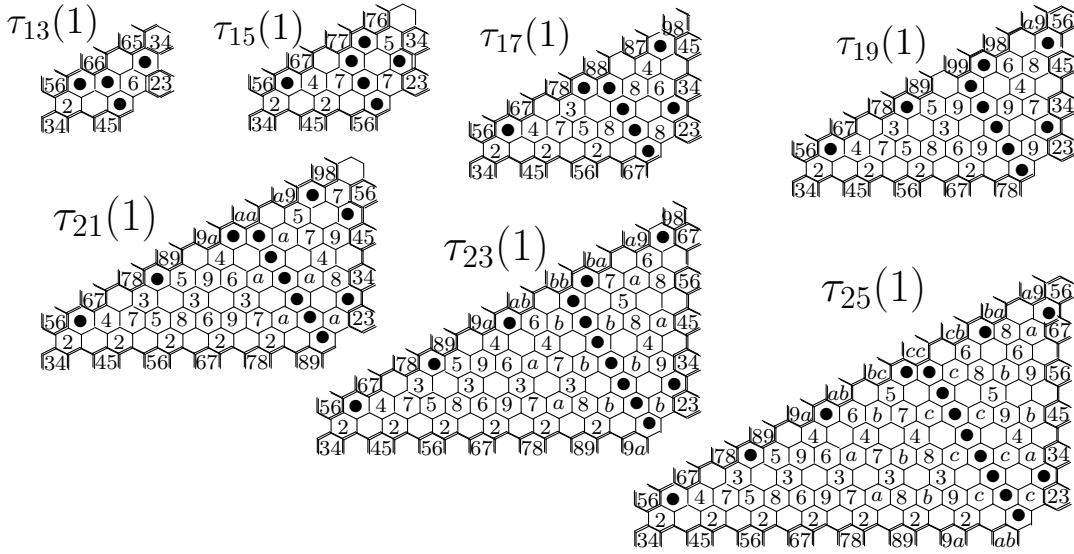


Figure 6: The representations $\tau_n(1)$, for $n = 13, \dots, 25$

For $n \geq 17$, the isolated vertices of $\sigma_n(1)$ are nonisolated in the other charts $\sigma_n(i)$, where $i \neq 1$ ranges over the units MOD n from 2 to $\lfloor n/2 \rfloor$. This suggests the following conjecture.

Conjecture 24 $G'_{n,4}$ is a connected graph, for $n \geq 17$.

On the other hand, the six charts $\tau_{13}(i)$, for $i = 1, \dots, 6$, represent the same pair of isolated vertices shown in Figure 1(a₁) and 1(a₂), which are thus the only components of $G'_{13,4}$. In addition, the four charts $\tau_{15}(i)$, for $i = 1, 2, 4, 7$, represent only a CT and 4 isolated vertices.

Proof. (of Theorem 1) The vertices $v \in V_6$ are exactly those represented by TMC K_4 -types in G . The four K_3 -types of such TMC K_4 -type form three distinct pairs of K_3 -types. Each of these three pairs corresponds to a triangle of G , yielding the triangles T_0, T_1, T_2 cited in the statement. Each pair $\{T_i, T_j\}$ with $i \neq j$ determines two different butterflies and thus two different charts $D_{i,j}^0$ and $D_{i,j}^1$. The difference $V(G) \setminus V_6$ is composed by vertices at distance < 3 both from the boundary straight-line paths of charts $\tau_n(i)$ and the diagonal paths $\eta(i)$ in charts $\tau_n(i)$ departing as in item (1) of Proposition 16 from boundary vertices realizing angles of 90° as in the upper right representation in Figure 4 and as in Figure 5. This insures

that $|V(G) \setminus V_6|$ grows linearly as n increases, while $|V_6|$ has a quadratic growth with respect to n . This proves that V_6 has asymptotically $|V(G)|$ vertices. Each one of the four K_3 -types composing the K_4 -type associated to a vertex of V_6 offer three positive integers that color the edges of a corresponding chart modeled on \mathcal{H} . Each of these three integers colors the edges of a parallel class of edges in that chart, as was presented in our previous work, [1]. These completes the proof of Theorem 1. \square

Proof. (of Theorem 2) Now, \mathcal{G}_1 must be restricted to the subfamily \mathcal{G}'_1 formed by the graphs $G_{n,4}$ for which n is an odd prime. This guarantees that the charts $\tau_n(i)$ are related with the tessellated graphs $D_{i,j}^k$ in the following way, for $i = 1, \dots, \frac{n}{2}$: Each chart $\tau_n(i)$ contains two components formed by vertices representing TMC K_4 -types. These components are: **(a)** contained in a 30° - 60° - 90° triangle R (formed by the three delimiting SA's); **(b)** separated by the path $\eta(i)$ in $\tau_n(i)$. The union of the two 30° - 60° - 90° triangles delimited by the SA's and $\eta(i)$ yields $\tau_n(i)$. There are $n/2$ charts $\tau_n(i)$, asymptotically of order $|V(G)|^{1/3}$, since each $\tau_n(i)$ has asymptotically $|V(G)|^{2/3}$ vertices. However, consider stripping bands of the delimiting SA's in the 30° - 60° - 90° triangles having a constant Euclidean width, in order to get away from any loops. This reduces the vertex number of such 30° - 60° - 90° triangles in just a multiple of their perimeters, which does not influence significantly in the said asymptotic vertex number. \square

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